lar cylinders, by McDonough[2] for spherical and cylindrical shells, by Raja and Vinson[7] for shallow spherical shells, and by Kliger and Vinson[6] for conical shells.

The given equations can also be reduced to those for classical, isotropic materials developed by Reissner[3] by neglecting transverse shear deformation, thermal strains, and the transverse isotropy of the material.

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Sawtooth Structure of a Convective Plasma Column

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Introduction

RECENT experimental results[1] have indicated that under some conditions the slanted column of the convective electric arc can move through sulfur hexafluoride at crossflow Mach numbers in significant excess of unity, ranging up to at least 3.8. On the other hand, observations[2,3] of the convective electric arc in air, as well as some observations in SF_6, indicate that the crossflow Mach number of the stable column is very nearly equal to unity.

It is shown herein that these apparently contradictory results can be reconciled through analysis of the fine, sawtooth structure that is shown to exist in the arc column in SF_6: the sawtooth column can move at an apparent crossflow Mach number greater than unity, and yet maintain an actual crossflow Mach number equal to unity. Photographic evidence for the existence of the sawtooth structure is discussed, two models for the sawtooth column are considered, favorable comparisons between theory and experiment are presented, and tentative deductions regarding the slanting mechanism are made. The experimental data discussed herein were obtained using the thermionic rail accelerator described in Ref. 3.

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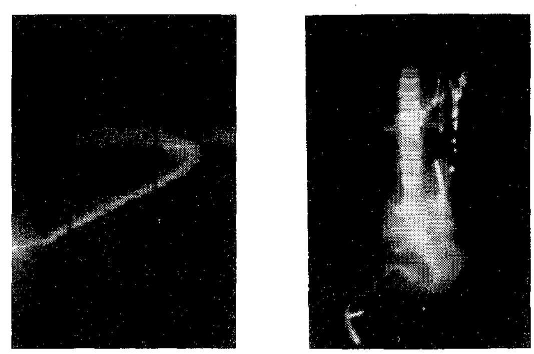
Experimental Indications of Column Structure in SF_6

The column of the electric arc moving through sulfur hexaflouride (or through air) slants across the electric field.[1–3] Figure 3a of Ref. 1 shows that in SF_6 the column slant angle θ can be considerably less than the slant of a Mach line.

In SF_6 the Mach slant is almost identical to the slant angle θ_E, which, for a given Mach number, results in a maximum in the ionization parameter, $(E_{||}/p_s)/(E_{||}/p_s)_{\theta=0}$. Here $E_{||}$ is the component of electric field parallel to the column, and p_s is the pressure at the leading boundary of the column.[2]

In cases where the slant is less than the Mach slant, that is, cases where the crossflow Mach number is greater than unity, the column seems to take on a sawtooth shape. At low Mach numbers the sawtooth shape is clearly visible. As the crossflow Mach number increases, the sawtooth angle becomes smaller, and the structure of the column becomes more difficult to discern. Structure is clearly present however, as seen for example in Fig. 1, where the crossflow Mach number M_c is 2.0.

In a number of photographs, plasma streamers were visible in the wake of the column.† These streamers seem to indicate that plasma is being spilled from the column at certain points, and that between these points, more plasma is being created than quenched. Such points appear from photographs to coincide with the trailing apices of the sawtooth column. The streamers are usually straight and perpendicular to the column, indicating in such cases that the over-all column is moving normal to itself, rather than parallel to the electrode rails. This means, incidentally, that plasma must be continually created at one electrode and quenched at the other,



a) b)

Fig. 1 Arc Columns in SF_6 at $M = 4.6$, $T_R = 1700°C$. a) Arc viewed from the side. The electric field is vertical; the direction of arc motion is from left to right; the incandescent carbon cathode rail is visible, the cold copper anode rail is not. The experimental conditions are: Mach number, 4.6; slant angle, 65°; crossflow Mach number, 2.0; ambient pressure, 20 torr; arc current, approximately 170 amp; magnetic induction, approximately 4.6 kgauss; interelectrode spacing, 7 cm; cathode preheat temperature, 1700°C. b) Arc viewed close up and head on. The experimental conditions are similar to those in a. The incandescent remnants of the firing wire visible in the photograph are far-removed from the arc column. The axis of the arc column lies in the electrode plane (vertical). Bright zones of light appear wherever the local column axis is more nearly parallel to the line of sight. The symmetrical horizontal zones of light illustrated here have been found, in simultaneous side-view end-view photographs, to correlate precisely with the steps of the sawtooth structure.

† The wake "streamers" are not to be confused with spark streamers.

a conclusion amply confirmed by observations of plasma laid along the cold rail by the arc column (cf. Fig. 1 of Ref. 1).

Model for the Sawtooth Column

Two possible shapes for the slanted sawtooth column are illustrated in Fig. 2. The type 1 sawtooth (Fig. 2a) has equal angles (crosscut saw). Sawtooth type 2 (Fig. 2b) has unequal angles (ripsaw). With either type, the straight line connecting the leading apices of the saw teeth is taken to be the leading edge of the sawtooth column; the angle between this line and the electric field is the column slant angle θ; and the velocity of this line normal to itself is the (apparent) column crossflow velocity V_c. The velocity of the arc in the direction of the electrode rails is given by

$$V = V_c/\cos\theta$$

Thus

$$\cos\theta = V_c/V = M_c/M$$

where M is the arc Mach number, and M_c is the (apparent) crossflow Mach number.

The sawtooth column is composed of straight segments, two for each sawtooth. The Lorentz force on each segment is normal to the segment and in the electrode plane. We shall assume that each column segment moves in the direction of the Lorentz force, at a constant velocity V_s. Thus V_s, the *actual* crossflow velocity of the local column segment, can be less than V_c, the *apparent* crossflow velocity of the sawtooth column.

For columns of type 1, the bisectors of the sawtooth angles are perpendicular to the leading edge of the column, defined previously. We note that for this type sawtooth, adjacent segments of the column do not slant across the electric field at angles of the same magnitude. The slanting of the type 1 segments must therefore be dependent on factors other than the electric field configuration. We also note that the sawteeth for the type 1 column will appear to move along the column as the column progresses down the electrode rails.

For the type 1 column, the sawtooth half angle α is given by

$$\sin\alpha = V_s/V_c = M_s/M_c$$

where M_s is the crossflow Mach number for the local segment. Substituting for M_c, we obtain

$$\cos\theta = M_s/(M \sin\alpha)$$

If the results of investigations of the (smooth) arc column in air are applicable, we may assume that M_s, the crossflow Mach number of the local segment of the column, is unity. This gives

$$\cos\theta = 1/(M \sin\alpha)$$

It will be shown below that this equation is borne out by experimental observations.

For columns of type 2, we assume the sawtooth bisectors to be parallel to the electrode rails (Fig. 2b). In this case, the magnitude of the angle at which the individual column segment slants across the electric field is the same for all segments. Thus the slanting of the type 2 column segments could be strongly dependent on the electric field configuration.

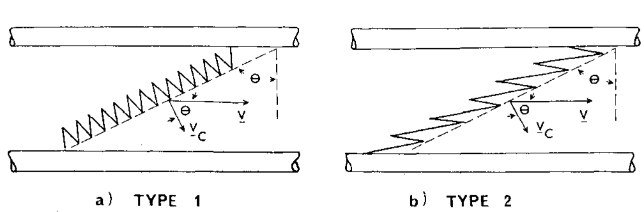


Fig. 2 Models for the sawtooth column.

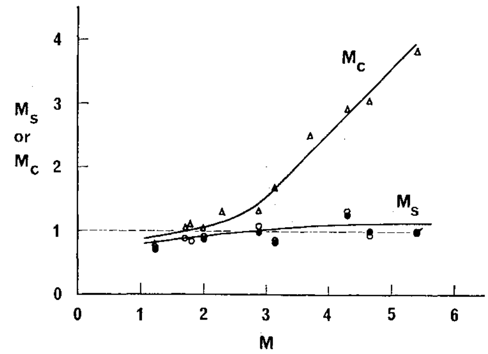


Fig. 3 Variation with Mach number of the apparent and actual crossflow Mach numbers in SF_6 for T_R = 1100°C.

We note that the sawteeth for the type 2 column do not move along the column.

For the type 2 column, the sawtooth half-angle β is given by

$$\sin\beta = V_s/V = M_s/M$$

Thus for the type 2 sawtooth column, the column slant angle is not related to the sawtooth angle. It will be shown below that the type 2 sawtooth column, adequately matches experimental observations, especially at high Mach numbers.

Discussion and Comparison with Experiment

The arc column seemed in photographs to approximate the type 2 sawtooth shape at the higher Mach numbers and the type 1 sawtooth at the lower Mach numbers, though it was often of some intermediate shape.

Using measured values of M, θ, and α or β for the sawtooth column, M_s was calculated from the preceding equations. The results are shown in Fig. 3. Also shown is the measured apparent crossflow Mach number, M_c (triangular symbols). It can be seen that whereas M_c increases almost linearly with M to a value near 4.0, M_s remains effectively equal to unity over the entire range in M. This result holds true whether the type 1 (circular symbols) or type 2 (solid symbols) sawtooth is assumed. For the convective arc in SF_6 as well as in air, there apprears to be some mechanism which results in an actual crossflow velocity approximately equal to the speed of sound.

At high Mach numbers, the slant angles given in Fig. 3a, Ref. 1, are considerably below the 90-μ curve, where the apparent crossflow Mach number is greater than unity. Calculations show that these data are well predicted by the sawtooth model with an actual crossflow Mach number of unity, despite uncertainties in the measurement of α

These models for the sawtooth column evidently require continual generation of new plasma at the leading apices and continual quenching, folding, or spilling of plasma at the trailing apices. No mechanism is here proposed to meet either of these requirements—except to refer, in the case of the trailing apices, to the plasma streamers discussed above, and also to the fact that SF_6 and its dissociation products are electrophilic.[4,5]

The existence and nature of the sawteeth are strong functions of the temperature of the heated rail. In many cases the sawteeth seem to move along the column from the heated rail toward the cold rail. It is of interest to note that if the type 1 sawtooth angle, α, is assumed to be equal to the Mach angle pertaining near the heated rail, then

$$\sin\alpha = 1/M_R$$

where M_R is the Mach number at the heated rail, and

$$\cos\theta = M_R/M = (T/T_R)^{1/2}$$

This equation predicts very well the actual column slant angles observed at heated-rail temperatures of 1700°C and 2200°C.[1]

The type 2 sawtooth column is associated with high arc Mach numbers. Though high Mach number arcs were observed which seemed not to have the type 2 sawtooth structure, only one type 2 arc was observed with a Mach number as low as 3.6, and all other type 2 arcs moved at Mach numbers above 4.5.

The appearance of the type 2 sawtooth at high Mach numbers could have one important implication. It could mean that the electric field plays a more important role in column slanting at high Mach numbers, since, as was noted earlier, the electric field can be important to the (equal) slanting of the segments of the type 2 column. This implication is supported by the fact that the peak in the ionization parameter is much sharper for $M \geqq 4.5$. However, these observations cannot be regarded as conclusive proof of an electric-field or ionization mechanism for slanting until more detailed studies of the column structure can be made. Changes in the nature of the finer structure could result from mutual interaction between adjacent sawtooth segments.

Conclusions

The following conclusions may be drawn: 1) There is a fine structure for the supersonic electric arc in sulfur hexafluoride. Side-view photographs show this structure to be such that the electric current of the moving arc column describes a slanted sawtooth path between electrodes. Front-view photographs show that the column axis does not spiral (generally) but remains in the electrode plane. Plasma streamers are sometimes visible in the wake of the sawtooth column, each streamer originating at a downstream apex of the sawtooth shape. 2) The sawtooth structure of the arc column in SF_6 results in an apparent crossflow Mach number which can be significantly higher than the actual crossflow Mach number for a local column segment. 3) Measurements indicate that for arc Mach numbers from about 1.5 to about 5.5, the actual crossflow Mach number is very nearly equal to unity. 4) Using an actual crossflow Mach number of unity, good agreement is obtained between the measured angle of slant of the arc column and that calculated from measurements of the sawtooth angle using the sawtooth-column theory. 5) The experimental observations indicate that the convective interaction mechanism which has been previously observed for the electric arc in air, and which results in a stable arc column slanted across the electric field lines such that the crossflow Mach number is near unity—the observations indicate that this mechanism is also locally effective for the electric arc in SF_6, and that in cases where transient conditions or root constraints dictate an arc with less slant, the arc column in SF_6 takes on a fine structure which nevertheless maintains the actual local crossflow Mach number near unity. 6) The apparent transition from the type 1 to the type 2 sawtooth at higher Mach numbers gives evidence of an ionization mechanism for column slanting, and indicates that there may in fact be two mechanisms for slanting, each of which tends to keep the actual crossflow Mach number near unity.

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Mach Disk in Underexpanded Exhaust Plumes

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Nomenclature

M = Mach number
p = pressure
r_j = jet exhaust radius
x = abscissa
y = ordinate (normal to centerline)
δ = streamtube width
γ = ratio of specific heats
ρ = density
τ = streamline slope = $\tan\theta$
θ = angle streamline makes with axis

Subscripts

SS = slipstream separating subsonic core streamtube from supersonic outer flow downstream of Mach disk
TP = triple point
j = conditions at jet exhaust plane
∞ = ambient conditions

THE basic features of the inviscid supersonic plume for static ambient are shown in Fig. 1. The expansion waves from the nozzle lip reflect from the constant pressure streamline as compression waves, subsequently coalescing to form the intercepting (barrel) shock. Depending on the flow conditions, the intercepting shock may reflect regularly at the centerline or it may terminate in a triple point-Mach disk configuration, illustrated in Fig. 1. Behind the Mach disk is a region of subsonic flow bounded above by a slipstream emanating from the triple point. This Note presents a flow model which explains in detail why a Mach disk is formed from the plume intercepting shock and, when it is formed,

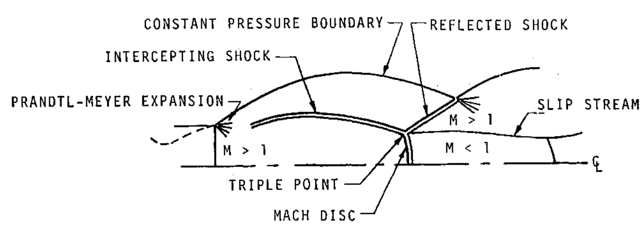

Fig. 1 Inviscid plume of underexpanded nozzle—static ambient.

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